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Some problems concerning to nilpotent lie superalgebras

J.R. Gómez [a,*], Yu. Khakimdjanov [b], R.M. Navarro [c]

[a] *Dpto. Matemática Aplicada I, Universidad de Sevilla, Sevilla, Spain*
[b] *Laboratoire de Mathématiques et Applications, Université de Haute Alsace, Mulhouse, France*
[c] *Dpto. de Matemáticas, Universidad de Extremadura, Cáceres, Spain*

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Abstract

The aim of this work is to present the first problems that appear in the study of nilpotent Lie superalgebras. These superalgebras and so the problems, will be viewed as a natural generalization of nilpotent Lie algebras.

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1. Introduction

In recent years, Lie superalgebra theory has had a profound effect on the evolution of mathematics and physics. The first comprehensive description of the mathematical theory of Lie superalgebras was given by Kac in 1977 [10]. Kac classified all simple Lie superalgebras with finite dimension over an algebraically closed field of characteristic zero. Semi-simple Lie superalgebras and their cohomology have been studied in [2,4,5,8,11,12].

Regarding papers concerning nilpotent Lie superalgebras, we have only found ones addressing their definition (using the descending central sequence as for Lie algebras) and Engel's Theorem. In 1998–1999, during a seminar at Haute Alsace University (Mulhouse, France), Professors Goze and Khakimdjanov proposed a program for generalizing the notion of filiform Lie algebras into Lie superalgebra theory.

* Corresponding author.
E-mail addresses: jrgomez@us.es (J.R. Gómez), y.hakimjanov@univ-mulhouse.fr (Yu. Khakimdjanov), rnavarro@unex.es (R.M. Navarro).

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For a nilpotent Lie algebra of dimension n the nilindex is evidently $\leq n-1$, and the Lie algebras with nilindex $n-1$ form the class of n-dimensional filiform Lie algebras [13].

For a nilpotent Lie superalgebra of type (n, m), the nilindex is always $\leq n+m-1$. The function $f(n, m)$, defined as the maximal nilindex for the Lie superalgebras of type (n, m), is, in general, not equal to $n+m-1$. Function determination is an open problem for the general case.

In this paper we show that $f(n, m) = n+m-1$ if and only if $n = 2$ and m is odd (Theorem 4.17). Moreover, for any odd m, there is only one Lie superalgebra, denoted by $K^{2,m}$, which verifies this condition, and the orbit of $K^{2,m}$ in the variety $\mathcal{N}^{2,m}$ of the Lie superalgebras of type $(2, m)$ is open for Zariski topology (Theorem 4.16). We also determine the function $f(n, m)$ for some other particular cases (Theorem 4.23).

We will refer to the nilpotent Lie superalgebras of type (n, m), with nilindex $f(n, m)$, as maximal class Lie superalgebras, and we will denote the variety of these Lie superalgebras as $\mathcal{M}^{n,m}$.

There is another generalization of the notion of the filiform Lie algebras for the case of Lie superalgebras. A nilpotent Lie superalgebra $\mathfrak{g}$ is called filiform if the even part, $\mathfrak{g}_0$, is a filiform Lie algebra (of dimension n) and the odd part, $\mathfrak{g}_1$, is a filiform $\mathfrak{g}_0$-module. We will denote this variety of Lie superalgebras by $\mathcal{F}^{n,m}$.

The study of the $\mathcal{F}^{n,m}$ class is simplified by the existence of “adapted” basis (see Theorem 3.5). In Gilg’s thesis [6], this theorem was used to investigate certain problems associated with filiform Lie superalgebras.

It is natural to ask what relationships exist between $\mathcal{M}^{n,m}$ and $\mathcal{F}^{n,m}$. At first, we quite naturally conjectured that $\mathcal{M}^{n,m} \subset \mathcal{F}^{n,m}$ (it is easy to show that $\mathcal{F}^{n,m} \not\subset \mathcal{M}^{n,m}$), but we show that, in fact, $\mathcal{M}^{n,m} \not\subset \mathcal{F}^{n,m}$ (see Theorem 6.1).

In Section 5 we give a classification for small dimensions. This classification was obtained as an illustration of the conjecture $\mathcal{M}^{2,m} \subset \mathcal{F}^{2,m}$ for any m odd.

We will not suppose any prior knowledge of the theory of Lie superalgebras. However, we do assume that the reader is familiar with the standard theory of Lie algebras. All the vector spaces that appear in this paper (and thus, all the algebras) are assumed to be $\mathbb{C}$-vector spaces with finite dimension.

2. Preliminaries

The vector space V is said to be $\mathbb{Z}_2$-graded if it admits a decomposition in direct sum, $V = V_0 \oplus V_1$. An element X of V is called homogeneous of degree γ ($\deg(X) = d(X) = \gamma$), $\gamma \in \mathbb{Z}_2$, if it is an element of V_γ. In particular, the elements of V_0 (resp. V_1) are also called even (resp. odd).

Let $V = V_0 \oplus V_1$ and $W = W_0 \oplus W_1$ be two graded vector spaces. A linear mapping $f : V \to W$ is said to be homogeneous of degree γ ($\deg(f) = d(f) = \gamma$), $\gamma \in \mathbb{Z}_2$, if $f(V_\alpha) \subset W_{\alpha+\gamma (\mathrm{mod}\, 2)}$ for all $\alpha \in \mathbb{Z}_2$. The mapping f is called a homomorphism of the $\mathbb{Z}_2$-graded vector space V into the $\mathbb{Z}_2$-graded vector space W if f is homogeneous of degree 0. Now it is evident how we define an isomorphism or an automorphism of $\mathbb{Z}_2$-graded vector spaces.

A superalgebra $\mathfrak{g}$ is just a $\mathbb{Z}_2$-graded algebra $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$. That is, if we denote by [,] the bracket product of $\mathfrak{g}$, we have $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta (\mathrm{mod}\, 2)}$ for all $\alpha, \beta \in \mathbb{Z}_2$.

Definition 2.1 (Scheunert [12]). Let $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ be a superalgebra whose multiplication is denoted by the bracket product [,]. We call $\mathfrak{g}$ a Lie superalgebra if the multiplication satisfies the following identities:

1. $[X, Y] = -(-1)^{\alpha\cdot\beta}[Y, X] \quad \forall X \in \mathfrak{g}_\alpha, \forall Y \in \mathfrak{g}_\beta$.
2. $(-1)^{\gamma\cdot\alpha}[X, [Y, Z]] + (-1)^{\alpha\cdot\beta}[Y, [Z, X]] + (-1)^{\beta\cdot\gamma}[Z, [X, Y]] = 0$ for all $X \in \mathfrak{g}_\alpha$, $Y \in \mathfrak{g}_\beta$, $Z \in \mathfrak{g}_\gamma$ with $\alpha, \beta, \gamma \in \mathbb{Z}_2$.

Identity 2 is called the graded Jacobi identity and it will be denoted by $J_g(X, Y, Z)$.

The *descending central sequence* of a Lie superalgebra $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ is defined by $\mathcal{C}^0(\mathfrak{g}) = \mathfrak{g}$, $\mathcal{C}^{k+1}(\mathfrak{g}) = [\mathcal{C}^k(\mathfrak{g}), \mathfrak{g}]$ for all $k \geq 0$. If $\mathcal{C}^k(\mathfrak{g}) = \{0\}$ for some k, the Lie superalgebra is called *nilpotent*. The smallest integer k such as $\mathcal{C}^k(\mathfrak{g}) = \{0\}$ is called the *nilindex* of $\mathfrak{g}$.

We define two new *descending sequences*, $\mathcal{C}^k(\mathfrak{g}_0)$ and $\mathcal{C}^k(\mathfrak{g}_1)$, as follows: $\mathcal{C}^0(\mathfrak{g}_i) = \mathfrak{g}_i$, $\mathcal{C}^{k+1}(\mathfrak{g}_i) = [\mathfrak{g}_0, \mathcal{C}^k(\mathfrak{g}_i)]$, $k \geq 0$, $i \in \{0, 1\}$.

If $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ is a nilpotent Lie superalgebra, then $\mathfrak{g}$ has super-nilindex or *s-nilindex* (p, q) [6], if the following conditions holds:

$$(\mathcal{C}^{p-1}(\mathfrak{g}_0))(\mathcal{C}^{q-1}(\mathfrak{g}_1)) \neq 0, \qquad \mathcal{C}^p(\mathfrak{g}_0) = \mathcal{C}^q(\mathfrak{g}_1) = 0.$$

Engel's theorem for Lie algebras and its direct consequences remain valid for Lie superalgebras, the proof being the same as for Lie algebras [3,9].

Engel's Theorem. *A lie superalgebra $\mathfrak{g}$ is nilpotent if and only if* $\mathrm{ad}_{\mathfrak{g}}(X)$ *is nilpotent for every homogeneous element X of $\mathfrak{g}$.*

Remark 2.2. It is known that if V is a vector space of dimension m and $\mathfrak{h}$ is a set of nilpotent endomorphism of V, then there exists a decreasing sequence of vector subspaces $V_m, \dots, V_1, V_0$ of V, with dimensions $m, m-1, \dots, 0$, respectively, and such that $h(V_{i+1}) \subseteq V_i \forall h \in \mathfrak{h}$ $i = 0, 1, \dots, m-1$. Thus, if we take a nilpotent Lie superalgebra, $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$, and consider $V = \mathfrak{g}_1$ ($\mathfrak{g}_1$ a vector space) and $\mathfrak{h}$ the operator ad restricted to $\mathfrak{g}_0$, we have a decreasing sequence of subspaces $V = V_m \supset \cdots \supset V_1 \supset V_0$ of dimensions $m, m-1, \dots, 0$, such that $[\mathfrak{g}_0, V_{i+1}] \subseteq V_i$.

We denote by $\mathcal{L}^{n,m}$ the set of the Lie superalgebras $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ with $\dim(\mathfrak{g}_0) = n$ and $\dim(\mathfrak{g}_1) = m$.

If we take an homogeneous basis $\{X_0, \dots, X_{n-1}, Y_1, \dots, Y_m\}$ for $\mathfrak{g}$ ($\mathfrak{g} \in \mathcal{L}^{n,m}$), the superalgebra is completely determined by its structure constants, that is, by the set of constants $\{C_{ij}^k, D_{ij}^k, E_{ij}^k\}_{i,j,k}$ that verify

$$[X_i, X_j] = \sum_{k=0}^{n-1} C_{ij}^k X_k, \quad 0 \leq i < j \leq n-1,$$

$$[X_i, Y_j] = \sum_{k=1}^{m} D_{ij}^k Y_k, \quad 0 \leq i \leq n-1, 1 \leq j \leq m,$$

$$[Y_i, Y_j] = \sum_{k=0}^{n-1} E_{ij}^k X_k, \quad 1 \leq i \leq j \leq m,$$

[,] being the bracket product of $\mathfrak{g}$. The structure constants of a Lie superalgebra verify the restrictions obtained by the graded Jacobi identities [7].

Let $V = V_0 \oplus V_1$ denote the underlying vector space of $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \in \mathcal{L}^{n,m}$ and $G(V)$ denote the group of the invertible linear maps of the form $f = f_0 + f_1$ such that $f_0 \in \mathrm{GL}(n, \mathbb{C})$ and $f_1 \in \mathrm{GL}(m, \mathbb{C})$ ($G(V) = \mathrm{GL}(n, \mathbb{C}) \oplus \mathrm{GL}(m, \mathbb{C})$). The action of $G(V)$ on $\mathcal{L}^{n,m}$ induces an action on the Lie superalgebra variety: two laws μ_1 and μ_2 are isomorphic, if there exists an $f = f_0 + f_1 \in G(V)$, such that

$$\mu_2(X, Y) = f_{\alpha+\beta}^{-1}(\mu_1(f_\alpha(X), f_\beta(Y))) \quad \forall X \in V_\alpha, \forall Y \in V_\beta.$$

We denote by $\mathcal{O}(\mu)$ the orbit of μ corresponding to this action.

3. Filiform Lie superalgebras $\mathcal{F}^{n,m}$

Next we consider filiform Lie superalgebras due to the fact that in this family the maximal class of nilindex $n + m - 1$ is included.

Definition 3.1. Any Lie superalgebra $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \in \mathcal{N}^{n,m}$ with s-nilindex $(n - 1, m)$ is called filiform.

We note by $\mathcal{N}_{p,q}^{n,m}$ the subset of $\mathcal{L}^{n,m}$ formed by all Lie superalgebras with s-nilindex (r, s), where $r \leq p$ and $s \leq q$.

Remark 3.2. We observe that the set $\mathcal{N}_{n-1,m}^{n,m}$ is the variety of all nilpotent Lie superalgebras. For simplicity we write $\mathcal{N}^{n,m}$ instead of $\mathcal{N}_{n-1,m}^{n,m}$.

We denote by $\mathcal{F}^{n,m}$ the subset of $\mathcal{N}^{n,m}$ composed of all filiform Lie superalgebras.

Definition 3.3. Let $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \in \mathcal{N}^{n,m}$. $\mathfrak{g}_1$ is called a $\mathfrak{g}_0$-filiform module if there exists a decreasing subsequence of vectorial subspaces in its underlying vectorial space V, $V = V_m \supset \cdots \supset V_1 \supset V_0$, with dimensions $m, m - 1, \dots, 0$, respectively, and such that $[\mathfrak{g}_0, V_{i+1}] = V_i$.

Corollary 3.4. *Let be $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \in \mathcal{F}^{n,m}$, then $\mathfrak{g}_0$ is a filiform Lie algebra and $\mathfrak{g}_1$ is a $\mathfrak{g}_0$-filiform module.*

Adapted basis. Prior to studying general classes of Lie (super)algebras it is convenient to solve the problem of finding a suitable basis; a so-called adapted basis. This question is not trivial for Lie superalgebras and it is very difficult to demonstrate the general existence of such a basis. We prove in the following theorem that there always exists an adapted basis for the class of filiform Lie superalgebras.

Theorem 3.5. *If $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \in \mathcal{F}^{n,m}$, then there exists an adapted basis of $\mathfrak{g}$, namely $\{X_0, X_1, \dots, X_{n-1}, Y_1, \dots, Y_m\}$, with $\{X_0, X_1, \dots, X_{n-1}\}$ a basis of $\mathfrak{g}_0$ and $\{Y_1, \dots, Y_m\}$*

a basis of $\mathfrak{g}_1$, *such that*:

$$[X_0, X_i] = X_{i+1}, \quad 1 \le i \le n-2, \qquad [X_0, X_{n-1}] = 0,$$
$$[X_0, Y_j] = Y_{j+1}, \quad 1 \le j \le m-1, \qquad [X_0, Y_m] = 0.$$

X_0 *is called the characteristic vector*.

Remark 3.6. This result was presented by the authors in 1999 during a seminar which took place in Colmar (Haute Alsace University, France) and was subsequently used by [6] in the study of low-dimensional filiform Lie superalgebras.

Proof of Theorem 3.5. As $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ is a filiform Lie superalgebra, then, in particular, $\mathfrak{g}_0$ is a filiform Lie algebra. Thus we have an adapted basis for $\mathfrak{g}_0 : \{X_0, X_1, \dots, X_{n-1}\}$ with $[X_0, X_i] = X_{i+1}, 1 \le i \le n-2$ and $[X_0, X_{n-1}] = 0$.

On the other hand, $\mathfrak{g}_1$ is a $\mathfrak{g}_0$-filiform module, so there exists a creasing subsequence of vectorial subspaces in the vectorial space corresponding to $\mathfrak{g}_1$, namely V_m, such that

$$0 \subset V_1 \subset \cdots \subset V_m \ \text{ with } \ \dim\left(\frac{V_{i+1}}{V_i}\right) = 1,$$

where each V_i will be the vectorial space of generators $\{Y_1, \dots, Y_i\}$, $V_i = \langle Y_1, \dots, Y_i\rangle$, with $[\mathfrak{g}_0, V_{i+1}] = V_i$.

By induction, it is possible to prove that there exists a set of non-null scalars namely $\{\lambda_{i_2}, \lambda_{i_3}, \dots, \lambda_{i_m}\}$ whose indices $\{i_2, i_3, \dots, i_m\} \subseteq \{0, 1, \dots, n-1\}$ verify

$$[X_{i_k}, Y_k] = \lambda_{i_k} Y_{k-1} + \Psi_k(Y_{k-2}, \dots, Y_1), \quad 2 \le k \le m,$$

where $\Psi_k(v_1, v_2, \dots, v_s)$ represents a linear combination of the vectors $\{v_1, v_2, \dots, v_s\}$.

Using the graded Jacobi identity we can assert that $\{i_2, \dots, i_m\} \subseteq \{0, 1\}$. Thus we have

$$[X_j, Y_1] = 0, \quad j = 1, 2,$$
$$[X_0, Y_2] = \lambda_2 Y_1, \qquad [X_1, Y_2] = \delta_2 Y_1 \quad (\lambda_2, \delta_2) \ne (0,0),$$
$$[X_0, Y_3] = \lambda_3 Y_2 + \Psi_3(Y_1), \qquad [X_1, Y_3] = \delta_3 Y_2 + \Phi_3(Y_1) \quad (\lambda_3, \delta_3) \ne (0,0),$$

$$[X_0, Y_i] = \lambda_i Y_{i-1} + \Psi_i(Y_{i-2}, \dots, Y_1), \quad 4 \le i \le m,$$
$$[X_1, Y_i] = \delta_i Y_{i-1} + \Phi_i(Y_{i-2}, \dots, Y_1) \quad (\lambda_m, \delta_m) \ne (0,0).$$

Using the change of basis $X'_0 = X_0 + \gamma X_1$, the new structure constants λ'_i are $\lambda'_i = \lambda_i + \gamma\delta_i$, and by choosing γ such that $\gamma \notin \{-\lambda_2/\delta_2, -\lambda_3/\delta_3, \dots, -\lambda_m/\delta_m\}$, λ'_i will be distinct from zero for all i.

An obvious new change of basis proves the theorem. □

4. The algebraic variety $\mathcal{N}^{n,m}$: Lie superalgebras of maximal nilindex $\mathcal{M}^{n,m}$

The aim of this paper is to determine the maximal class. It is easy to see that if $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \in \mathcal{N}^{n,m}$ with nilindex $n + m - 1$, then $\mathfrak{g}_0$ is a filiform Lie algebra and $\mathfrak{g}_1$ is a

$\mathfrak{g}_0$-filiform module; that is $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \in \mathcal{F}^{n,m}$ and $\mathcal{M}^{n,m} \subset \mathcal{F}^{n,m}$ in this case. However, in this section we will prove that this nilindex is only possible for $n = 2$ and m odd. In all the cases that remain, the maximal nilindex (or the nilindex of the maximal class $\mathcal{M}^{n,m}$) will be $\leq n + m - 2$ and then we have in general

$$\mathcal{M}^{n,m} \neg \subset \mathcal{F}^{n,m},$$

which complicates the issue because no adapted basis exist outside $\mathcal{F}^{n,m}$.

Next we consider some of the properties of the algebraic subvariety $\mathcal{N}^{n,m}$.

Proposition 4.1. *$\mathcal{N}^{n,m}_{p,q}$ is an algebraic subvariety of $\mathcal{L}^{n,m}$.*

Proof. The set $\mathcal{N}^{n,m}_{p,q}$ of $\mathcal{L}^{n,m}$ is defined by the restrictions $\mathcal{C}^p(\mathfrak{g}_0) = 0$ and $\mathcal{C}^q(\mathfrak{g}_1) = 0$, but these restrictions are polynomial equations of the structure constants. Thus $\mathcal{N}^{n,m}_{p,q}$ is closed for Zariski topology and it will have the structure of an algebraic subvariety. We denote by $\mathcal{N}^{n,m}_{p,q}$ the corresponding affine variety. □

We denote by $\mathcal{N}^{n,m}_k$ the subset of $\mathcal{L}^{n,m}$ formed by all Lie superalgebras with nilindex less or equal to k.

Proposition 4.2. *$\mathcal{N}^{n,m}_k$ is an algebraic subvariety of $\mathcal{L}^{n,m}$.*

Proof. The set $\mathcal{N}^{n,m}_k$ of $\mathcal{L}^{n,m}$ is defined as follows:

$$\mathcal{N}^{n,m}_k = \{\mu \in \mathcal{L}^{n,m} / \mu(X_1, \mu(X_2, \ldots, \mu(X_k, X_{k+1})) \cdots) = 0\}$$

for any $X_1, \ldots, X_{k+1} \in \mathbb{C}^{n,m}$.

It is easy to see that, if we fix a basis of $\mathbb{C}^{n,m}$, the laws of Lie superalgebras are identified with their structure constants that verify polynomial relations as was seen in Section 2. The set $\mathcal{N}^{n,m}_k$ given by the above definition will come done by polynomial relations too, so $\mathcal{N}^{n,m}_k$ is closed for Zariski topology and it will have the structure of an algebraic subvariety of $\mathcal{L}^{n,m}$. We denote by $\mathcal{N}^{n,m}_k$ the corresponding affine variety. □

Remark 4.3. We observe that the set $\mathcal{N}^{n,m}_{n+m-1}$ is the variety of all the nilpotent Lie superalgebras $\mathcal{N}^{n,m}$. Thus we have

$$\mathcal{N}^{n,m}_{n+m-1} = \mathcal{N}^{n,m} = \mathcal{N}^{n,m}_{n-1,m}.$$

Definition 4.4. A Lie superalgebra $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \in \mathcal{N}^{n,m}$ is said to be of maximal nilindex, or belongs to the maximal class, if its nilindex is the maximum possible.

Remark 4.5. The problem of finding the maximal nilindex for any pair of dimensions n and m (dimensions of the even and odd parts, respectively) is still an open one. The function that gives the maximal nilindex for each pair of dimensions n and m will be noted by $f(n, m)$. For nilpotency it is easily see that $f(n, m) \leq n + m - 1$.

We note by $\mathcal{M}^{n,m}$ the maximal class composed of all the nilpotent Lie superalgebras of maximal nilindex $f(n, m)$. By construction $\mathcal{M}^{n,m}$ is the true generalization of the filiform

Lie algebras in the theory of Lie superalgebras, not the filiform Lie superalgebras, $\mathcal{F}^{n,m}$, presented in [6].

Proposition 4.6. *Each component of $\mathcal{M}^{n,m}$ determines a component of $\mathcal{N}^{n,m}$.*

Proof. $\mathcal{M}^{n,m} = \mathcal{N}^{n,m} - \mathcal{N}^{n,m}_{f(n,m)-1}$ is a Zariski open subset of $\mathcal{N}^{n,m}$. □

Corollary 4.7. *For any $\mu \in \mathcal{M}^{n,m}$ the Zariski closure of the orbit $\mathcal{O}(\mu)$, $\overline{\mathcal{O}(\mu),}^{\mathcal{Z}}$ will be an irreducible component of $\mathcal{N}^{n,m}$.*

As we have already noted in general there is not an adapted basis, so we can only speak in terms of an adequate basis. Thus we have the following lemma.

Lemma 4.8. *If $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \in \mathcal{N}^{n,m}$, then there exists a basis $\{X_0, X_1, \dots, X_{n-1}, Y_1, \dots, Y_m\}$* of $\mathfrak{g}$, *with $\{X_0, X_1, \dots, X_{n-1}\}$ a basis of $\mathfrak{g}_0$ and $\{Y_1, \dots, Y_m\}$ a basis of $\mathfrak{g}_1$, such that*:

$$[X_0, X_i] = \varepsilon_i X_{i+1}, \quad 1 \le i \le n-2, \qquad [X_0, X_{n-1}] = 0,$$
$$[X_0, Y_j] = \delta_j Y_{j+1} + (1-\delta_j)\Psi_j(Y_{j+2}, \dots, Y_m), \quad 1 \le j \le m-1,$$
$$[X_0, Y_m] = 0, \quad \varepsilon_i, \delta_j \in \{0, 1\},$$

where X_0 is just a characteristic vector of the Lie algebra $\mathfrak{g}_0$.

Proof. We take an adapted basis for $\mathfrak{g}_0$ (such a basis exists for $\mathfrak{g}_0$ be a nilpotent Lie algebra) and take also, as a basis of $\mathfrak{g}_1$, the vectors corresponding to the decreasing sequence of vectorial subspaces of $\mathfrak{g}_1$ (this exists for $\mathfrak{g}_1$ be a $\mathfrak{g}_0$-module). Then, the result is obtained applying simple changes of basis as and when required. □

The first question we address is whether there exist Lie superalgebras with maximal nilindex $f(n,m) = n+m-1$ and thus if there exists a pair of values (n,m) for which this maximal nilindex is obtained. It is natural to perform this search in $\mathcal{F}^{n,m}$, but such a strategy cannot be employed for cases in which the function $f(n,m) \le n+m-2$.

Searching a superalgebra as described above we find $K^{2,m}$.

Example 4.9 (Of existence). In what follows we denote by $K^{2,m}$ (m odd) the family of Lie superalgebras for which the products in an adapted basis $\{X_0, X_1, Y_1, \dots, Y_m\}$ are:

$$K^{2,m}: \begin{cases} [X_0, Y_i] = Y_{i+1}, & 1 \le i \le m-1, \\ [Y_i, Y_{m+1-i}] = (-1)^{(i+1)} X_1, & 1 \le i \le \frac{1}{2}(m+1). \end{cases}$$

$K^{2,m}$ (m odd) is a family of Lie superalgebras with maximal nilindex $n+m-1 = m+1$. So, $f(2,m) = m+1$ for m odd.

Remark 4.10. $\mathcal{O}(K^{2,m})$ will be a family of Lie superalgebras of maximal nilindex $m+1$.

Next we study in more detail this orbit, obtaining that it is an open set for the Zariski topology in $\mathcal{N}^{2,m}$ for any m odd. This result is important because if the orbit is open then its dimension coincides with the dimension of its Zariski closure, and this closure is a component of the variety of nilpotent Lie superalgebras. We first need the following two lemmas.

Lemma 4.11. *Suppose* $\mathfrak{g} \in \mathcal{F}^{2,m}$, *with* m *odd and with adapted basis* $\{X_0, X_1, Y_1, \dots, Y_m\}$. *If the structure constant* $E^1_{1m} \neq 0$ *then* $\mathfrak{g}$ *belongs to the following family of filiform Lie superalgebras* $\mu(\alpha_1, \dots, \alpha_{(m-1)/2})$ *defined by*

$$[X_0, Y_i] = Y_{i+1}, \quad 1 \leq i \leq m-1,$$
$$[Y_i, Y_{m+1-i}] = (-1)^{i+1} X_1, \quad 1 \leq i \leq \tfrac{1}{2}(m+1),$$
$$[Y_i, Y_{2k-i}] = (-1)^{i+1} \alpha_k X_1, \quad 1 \leq i \leq k,\ 1 \leq k \leq \tfrac{1}{2}(m-1)$$

with $\alpha_k \in \mathbb{C}$ *for* $1 \leq k \leq (m-1)/2$.

Remark 4.12. $E^1_{1m} \neq 0$ signifies that $\dim(\mathcal{Z}(\mathfrak{g})) = 1$.

Proof. Let $\mathfrak{g}$ be as described under the assumptions of the lemma and with an adapted basis $\{X_0, X_1, Y_1, \dots, Y_m\}$. If $E^1_{1m} \neq 0$ without loss of generality we can suppose that $E^1_{1m} = 1$ and then, the descending central sequence is

$$\mathcal{C}^i(\mathfrak{g}) = \langle X_1 \rangle \oplus \langle Y_{i+1}, \dots, Y_m \rangle, \quad 1 \leq i \leq m-1, \qquad \mathcal{C}^m(\mathfrak{g}) = \langle X_1 \rangle \oplus \langle 0 \rangle,$$
$$\mathcal{C}^j(\mathfrak{g}) = \langle 0 \rangle \oplus \langle 0 \rangle, \quad j > m+1.$$

By induction and appropriate use of the graded Jacobi identity we obtain the result. □

Remark 4.13. The family $\mu(\alpha_1, \dots, \alpha_{(m-1)/2})$, with m odd, is open in $\mathcal{F}^{2,m}$ for Zariski induced topology. As $\mathcal{F}^{2,m}$ is open in $\mathcal{N}^{2,m}$ then $\mu(\alpha_1, \dots, \alpha_{(m-1)/2})$ is open in $\mathcal{N}^{2,m}$.

Remark 4.14. In the family of the previous lemma we obtain $K^{2,m}$ for $\alpha_k = 0$, $1 \leq k \leq (m-1)/2$.

Lemma 4.15. *For any odd* m, *the* $((m-1)/2)$*-parametric family* $\mu(\alpha_1, \dots, \alpha_{(m-1)/2})$ *is included in the orbit of* $K^{2,m}$

$$\mu(\alpha_1, \dots, \alpha_{(m-1)/2}) \subseteq \mathcal{O}(K^{2,m}).$$

Proof. Let $\mathfrak{g}$ be any Lie superalgebra of the family $\mu(\alpha_1, \dots, \alpha_{(m-1)/2})$.

Making the change of basis defined by

$$X'_0 = X_0, \qquad X'_1 = a_1^2 X_1, \quad a_1 \neq 0, \qquad Y'_1 = \sum_{i=1}^{m} a_i Y_i,$$

$$Y'_j = [X'_0, Y'_{j-1}] = \sum_{i=1}^{m-j+1} a_i Y_{i+j-1}, \quad 2 \leq j \leq m,$$

where a_i, $2 \leq i \leq m$, are free parameters, the new structure constant E^1_{1m} remains equal to 1. For the bracket products

$$[Y_i, Y_{m-1-i}] = (-1)^{i+1}\alpha_{(m-1)/2}X_1, \quad 1 \leq i \leq \tfrac{1}{2}(m-1),$$

we obtain that $\alpha'_{(m-1)/2}$ is a polynomial equation in a_2 of degree 2. It is possible to select a_2 as one of the roots of $\alpha'_{(m-1)/2}$. Similarly, and by induction, we obtain the theorem. □

Theorem 4.16. *For any odd m there exists an orbit in $\mathcal{N}^{2,m}$, namely $\mathcal{O}(K^{2,m})$, that is open for Zariski topology.*

Proof. The result follows from $\mathcal{O}(K^{2,m}) \subseteq \mu(\alpha_1, \ldots, \alpha_{(m-1)/2})$. In fact, if $\mathfrak{g} \in \mathcal{O}(K^{2,m})$ then $\dim(\mathcal{Z}(\mathfrak{g})) = 1$, so the result is obtained because, if not, $\dim(\mathcal{Z}(\mathfrak{g})) = 2$ *which is a contradiction*. □

Theorem 4.17 (Main theorem for maximal class). *The only non-trivial Lie superalgebra, up to isomorphism, with nilindex $n + m - 1$ is $K^{2,m}$ which occurs for $n = 2$ and m odd.*

Before proving the theorem, the following proposition is required.

Proposition 4.18. *The maximal nilindex of the family $\mathcal{F}^{n,m}$ is $n + m - 1$ and it is only obtained for $n = 1$ (degenerated case) and for $n = 2$ and m odd.*

Proof. The proof is very laborious and needs the three following technical lemmas that are proved using the graded Jacobi identity.

Lemma 4.19. *Suppose $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \in \mathcal{F}^{n,m}$. If the vector X_k appears in the bracket $[Y_1, Y_m]$ and $X_l \notin B(\mathfrak{g}_1 \times \mathfrak{g}_1)$ with $0 \leq l < k$ (according to the notation of [1]) then m is odd and $k = n - 1$.*

Lemma 4.20. *Suppose $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \in \mathcal{F}^{n,m}$. If the vector X_1 appears in the bracket $[Y_i, Y_j]$ for the case $n \geq 3$ and m odd then $i + j$ is even and $2 \leq i + j \leq m - 1$.*

Lemma 4.21. *Suppose be $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \in \mathcal{F}^{n,m}$. If the vector X_1 appears in the bracket $[Y_i, Y_j]$ in the case $n \geq 3$ and m even then $i + j$ is even and $2 \leq i + j \leq m$.*

In order to prove the proposition it is necessary to consider the following cases separately: (i) $n = 1$, (ii) $m = 1$, (iii) $n \geq 3$ and m *odd*, (iv) $n \geq 3$ and m *even* and (v) $n = 2$ and m *even*.

The proof for case (i) is trivial. The result for case (ii) is obtained if adequate graded Jacobi identities are considered. The result for case (iii) is obtained using Lemma 4.20 and considering all the possible bracket products $[Y_i, Y_i]$ with $1 \leq i \leq (m-1)/2$, in which vector X_1 could appear. Cases (iv) and (v) are analogous, using Lemma 4.21 instead of Lemma 4.20.

Proof of Theorem 4.17. It is easy to see that the condition of nilindex $n+m-1$ leads one to consider filiform Lie superalgebras (this fact is not true for nilindices $\leq n+m-2$). Given the previous proposition and the structure of the orbit of $K^{2,m}$ determined in $\mathcal{F}^{2,m}$ by the condition $E_{1m}^{1} \neq 0$, it only remains to determine if the structure constant $E_{1m}^{1}=0$. Then by the graded Jacobi identity, we never can obtain maximal nilindex $m+1$ $(n+m-1)$. □

Corollary 4.22. *For $n=2$ and m odd the conjecture $\mathcal{M}^{n,m} \subset \mathcal{F}^{n,m}$, is true; that is, $\mathcal{M}^{2+m} \subset \mathcal{F}^{2+m}$. Further,*

$$\mathcal{M}^{2+m} = \mathcal{O}(K^{2,m}) = \mu(\alpha_1, \ldots, \alpha_{(m-1)/2}).$$

For $n=2$ and any even m we have solved the problem of finding the maximal nilindex function, $f(n,m)$, and determining the maximal class, $\mathcal{M}^{n,m}$, or set of nilpotent Lie superalgebras, with nilindex equal to $f(n,m)$. Further, for all the remaining possibilities for the pair of dimension (n,m), we have proved that

$$f(n,m) \leq n+m-2.$$

Now, one can find lots of cases for (n,m) where the function $f(n,m)$ takes the value $n+m-2$. Thus, we have the following theorem.

Theorem 4.23. *The maximal nilindex function, $f(n,m)$, is equal to $n+m-2$ if n and m belong to one of the following cases*:

(i) $n=2$ *and* m *even.*
(ii) $n=3$.
(iii) $n=4$ *and* $m=5$ *or* m *even.*
(iv) $n \in \{m, m+1\}$.
(v) $n=m+2$ *with* m *even.*

Remark 4.24. The case $m=1$ and $n=1$ are trivial and we obtain $f(1,1)=2=f(2,1)$, $f(n,1)=n-1$ for $n \geq 3$ and $f(1,m)=m$.

Proof of Theorem 4.23. It is enough to present in each case a family of Lie superalgebras with nilindex $n+m-2$. In some cases the work needed to obtain these families has been very laborious, so we will only give the final expression.

- For the first case (i) we can consider the family of Lie superalgebras $K^{2,m}$, with m even, defined by

$$[X_0, Y_i] = Y_{i+1}, \quad 1 \leq i \leq m-1,$$
$$[Y_i, Y_{m-i}] = (-1)^{(m-2i)/2} X_1, \quad 1 \leq i \leq \tfrac{1}{2}m.$$

  (We have used the same notation, $K^{2,m}$, because it is the natural adaptation of $K^{2,m}$ (m odd) to the case when m is even.)

- For $n = 3$ and m odd, we can consider $K^{3,m}$, the family of laws that can be expressed in an adapted basis $\{X_0, X_1, X_2, Y_1, Y_2, \dots, Y_m\}$ by

$$[X_0, X_1] = X_2, \qquad [X_0, Y_i] = Y_{i+1}, \quad 1 \le i \le m-1,$$
$$[Y_i, Y_{m+1-i}] = (-1)^{i+1} X_2, \quad 1 \le i \le \tfrac{1}{2}(m+1).$$

If $n = 3$ and m is even we can consider the following family:

$$[X_0, X_1] = X_2, \qquad [X_0, Y_i] = Y_{i+1}, \quad 1 \le i \le m-1,$$
$$[Y_i, Y_{m-i}] = (-1)^{(m-2i)/2} X_1, \quad 1 \le i \le \tfrac{1}{2}m,$$
$$[Y_i, Y_{m+1-i}] = (-1)^{(m-2i)/2}(\tfrac{1}{2}(m-2i+1))X_2, \quad 1 \le i \le \tfrac{1}{2}m.$$

- For $n = 4$ and m even, we can consider the following family of laws that can be expressed in an adapted basis $\{X_0, X_1, X_2, X_3, Y_1, Y_2, \dots, Y_m\}$ by

$$[X_0, X_i] = X_{i+1}, \quad 1 \le i \le 2, \qquad [X_0, Y_i] = Y_{i+1}, \quad 1 \le i \le m-1,$$
$$[Y_i, Y_{m-i}] = (-1)^{(m-2i)/2} X_1, \quad 1 \le i \le \tfrac{1}{2}m,$$
$$[Y_i, Y_{m+1-i}] = (-1)^{(m-2i)/2}(\tfrac{1}{2}(m-2i+1))X_2, \quad 1 \le i \le \tfrac{1}{2}m,$$
$$[Y_i, Y_{m+2-i}] = (-1)^{(m-2i+2)/2}(\tfrac{1}{2}((i-1)m-(i-1)^2))X_3, \quad 2 \le i \le \tfrac{1}{2}(m+2).$$

If $n = 4$ and $m = 5$ we can consider the following superalgebra:

$$[X_0, X_i] = X_{i+1}, \quad 1 \le i \le 2, \qquad [X_0, Y_i] = Y_{i+1}, \quad 1 \le i \le 4,$$
$$[X_1, Y_3] = Y_5, \qquad [X_2, Y_2] = -Y_5, \qquad [X_3, Y_1] = Y_5,$$
$$[Y_1, Y_3] = -X_1, \qquad [Y_1, Y_4] = -\tfrac{3}{2}X_2, \qquad [Y_2, Y_2] = X_1, \qquad [Y_2, Y_3] = \tfrac{1}{2}X_2,$$
$$[Y_2, Y_4] = -\tfrac{3}{2}X_3, \qquad [Y_3, Y_3] = 2X_3.$$

- Cases for which $n = m$ and $n = m + 1$. For these cases it is enough to consider the following family of laws whose products can be expressed in an adapted basis $\{X_0, X_1, \dots, X_n, Y_1, \dots, Y_m\}$ by

$$[X_0, X_i] = X_{i+1}, \quad 1 \le i \le n-2, \qquad [X_0, Y_j] = Y_{j+1}, \quad 1 \le j \le m-1,$$
$$[X_1, X_k] = -X_{k+1}, \quad 2 \le k \le n-2, \qquad [X_1, Y_r] = -Y_{r+1}, \quad 2 \le r \le m-1,$$
$$[X_r, Y_1] = Y_{r+1}, \quad 2 \le r \le m-1, \qquad [Y_1, Y_1] = X_1,$$
$$[Y_1, Y_r] = \tfrac{1}{2}X_r, \quad 2 \le r \le n-1.$$

These families have nilindex $n + m - 2$.

- The case $n = m + 2$, m even. In this case we consider the Lie superalgebra of nilindex $n + m - 2$ that can be expressed in an adapted basis $\{X_0, X_1, \dots, X_{m+1}, Y_1, \dots, Y_m\}$ by

$$[X_0, X_i] = X_{i+1}, \quad 1 \le i \le m, \qquad [X_0, Y_j] = Y_{j+1}, \quad 1 \le j \le m-1,$$
$$[X_1, X_k] = -X_{k+1}, \quad 2 \le k \le m-1,$$
$$[X_i, X_{m+1-i}] = (-1)^{i+1} X_{m+1}, \quad 2 \le i \le \tfrac{1}{2}m,$$

$$[X_1, Y_j] = -Y_{j+1}, \quad 2 \le j \le m-1,$$
$$[X_j, Y_1] = Y_{j+1}, \quad 2 \le j \le m-1, \qquad [Y_1, Y_1] = X_1,$$
$$[Y_1, Y_k] = \tfrac{1}{2}X_k, \quad 2 \le k \le m,$$
$$[Y_i, Y_{m+2-i}] = (-1)^i \tfrac{1}{2} X_{m+1}, \quad 2 \le i \le \tfrac{1}{2}(m+2). \qquad \square$$

Proposition 4.25. *In the filiform case it is verified that, if* $n \ge 2m+1$, *then*

$$f(n, m) = n - 1.$$

Proof. By induction, and using the graded Jacobi identity, we obtain $X_i \notin \mathrm{Im}(\mathrm{ad}\, Y_i, \dots, \mathrm{ad}\, Y_m)$ with $1 \le i \le m$.

Consequently the descending central sequence is

$$\mathcal{C}^i(\mathfrak{g}) = \langle X_{i+1}, \dots, X_{n-1}\rangle \oplus \langle Y_{i+1}, \dots, Y_m\rangle, \quad 1 \le i \le m-1,$$
$$\mathcal{C}^i(\mathfrak{g}) = \langle X_{i+1}, \dots, X_{n-1}\rangle \oplus \langle 0\rangle, \quad m \le i \le n-2, \qquad \mathcal{C}^{n-1}(\mathfrak{g}) = \langle 0\rangle,$$

and the nilindex is $n-1$. □

Corollary 4.26. *In general, we have that* $f(n, m) \ge n-1$ *if* $n \ge 2m+1$.

We *conjecture* that in the general case the maximal nilindex function $f(n, m)$ stabilizes at $n-1$ as in the filiform case.

5. Classifications for specific dimensions

The classifications that we present in this section are illustrations of the validity of the conjecture $\mathcal{M}^{2,m} \not\subset \mathcal{F}^{2,m}$ for any odd m. Thus, we have the following results.

Proposition 5.1. *Let* $\mathfrak{g}$ *be any Lie superalgebra* $\mathfrak{g} \in \mathcal{N}^{2,1}$. *Then it is isomorphic to one of the two following Lie superalgebras that can be expressed in an adapted basis* $\{X_0, X_1, Y\}$ *by*

$-\mu_{2,1}$ (*abelian*), $\qquad -K^{2,1} : [Y, Y] = X_1$.

Proof. The proof is trivial. □

Theorem 5.2 (Classification). *Let* $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ *be any Lie superalgebra* $\mathfrak{g} \in \mathcal{N}^{2,3}$. *If* $\mathfrak{g}$ *is not a filiform Lie superalgebra then it is isomorphic to one of the following Lie superalgebras that can be expressed in an adapted basis* $\{X_0, X_1, Y_1, Y_2, Y_3\}$ *by*

$-\mu_{2,3}$ (*abelian*), $\qquad -B^{2,3} : [Y_i, Y_j] = b_{ij}^0 X_0 + b_{ij}^1 X_1$ ($\mathfrak{g}_1 =$ *trivial* $\mathfrak{g}_0$*-module*),

$-L_2 \oplus \mathbb{C}^2 : [X_0, Y_1] = Y_2$ (L_2 *filiform Lie algebra*),

$-\mu^1 (\simeq \mu_2 \oplus_s \mathbb{C}^3) : [X_0, Y_1] = Y_2, [X_1, Y_1] = Y_3$ (*metabelian Lie algebra*),

$-\mathcal{H}_2 (\simeq \mu_2 \oplus_s \mathbb{C}^3) : [X_0, Y_1] = Y_3$ (*Heisenberg Lie algebra*),

$-\mu^2 : [X_0, Y_1] = Y_2, [Y_1, Y_1] = X_1 \quad (\textit{split case}\, \mathcal{N}^{2,2} \oplus \mathbb{C}),$
$-\mu^3 : [X_0, Y_1] = Y_3, [X_1, Y_2] = -Y_3, [Y_1, Y_1] = X_1, [Y_1, Y_2] = \frac{1}{2} X_0,$
$-\mu^4 : [X_0, Y_1] = Y_2, [Y_1, Y_1] = X_1, [Y_3, Y_3] = X_1,$
$-\mu^5 : [X_0, Y_1] = Y_2, [Y_1, Y_3] = X_1, \qquad -\mu^6 : [X_0, Y_1] = Y_2, [Y_3, Y_3] = X_1.$

Remark 5.3. The Lie superalgebras $\mu_{2,3}$, $B^{2,3}$, $L_2 \oplus \mathbb{C}^2$, μ^1, $\mathcal{H}_2$ will be considered as *degenerate cases*.

Remark 5.4. There are only four, up to isomorphism, Lie superalgebras non-degenerate and non-split for the variety $\mathcal{N}^{2,3} - \mathcal{F}^{2,3}$. These Lie superalgebras are μ^i with $3 \le i \le 6$.

Proof. It is clear that all these Lie superalgebras are in $\mathcal{N}^{2,3} - \mathcal{F}^{2,3}$. It is trivial to show that any degenerate case is not isomorphic to any another.

For the non-degenerate cases it is sufficient to show

	nilindex	$\dim(\mathcal{C}\mathrm{ent}_{\mathfrak{g}}(\mathfrak{g}_1))$	$\dim[\mathcal{C}\mathrm{ent}_{\mathfrak{g}_1}(\mathfrak{g}_0), \mathcal{C}\mathrm{ent}_{\mathfrak{g}_1}(\mathfrak{g}_0)]$
μ^3	3	–	–
μ^4	2	2	1
μ^5	2	2	0
μ^6	2	3	–

It only remains to prove that there are no other possibilities. From Theorem 4.7, there exists an homogeneous basis $\{X_0, X_1, Y_1, Y_2, Y_3\}$ such that

$$[X_0, Y_1] = \delta_1 Y_2 + (1 - \delta_1)\Psi_1(Y_3), \qquad [X_0, Y_2] = \delta_2 Y_3$$

with $\delta_1, \delta_2 \in \{0, 1\}$. There are six possibilities for $[X_0, Y_i]$, $1 \le i \le 2$ and there are different cases for each one of them. However, using, essentially, graded Jacobi identities and adequate change of basis we obtain the result. □

6. Relations between $\mathcal{M}^{n,m}$ and $\mathcal{F}^{n,m}$

It is easy to prove that $\mathcal{F}^{n,m} \not\subset \mathcal{M}^{n,m}$. For example, if we consider any Lie superalgebra $\mathfrak{g} \in \mathcal{F}^{2,m} - \mathcal{O}(K^{2,m})$, with m odd, the structure constant E^1_{1m} is equal to zero which implies that $\mathfrak{g}$ will be of nilindex m which is not maximal (the maximal nilindex in this case is $m+1$). So $\mathfrak{g}$ will be a filiform Lie superalgebra that is not in the maximal class.

Finally, in this section we prove that $\mathcal{M}^{n,m} \not\subset \mathcal{F}^{n,m}$.

We use the family of non-filiform Lie superalgebras of $\mathcal{M}^{3,m}$, with m odd, which can be expressed in an adapted basis $\{X_0, X_1, X_2, Y_1, \dots, Y_m\}$ by the products

$$[X_0, Y_i] = Y_{i+1}, \quad 1 \le i \le m-1,$$
$$[Y_j, Y_{2k-j}] = (-1)^j (X_1 - kX_2), \quad 1 \le j \le k, 1 \le k \le \tfrac{1}{2}(m+1).$$

Theorem 6.1. $\mathcal{M}^{n,m} \not\subset \mathcal{F}^{n,m}$.

Proof. The mentioned family has maximal nilindex $m+1$ $(= n+m-2)$ for each dimension m which is odd. It is clearly not a filiform Lie superalgebra because the even part is abelian and hence not filiform, which proves the theorem. □

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